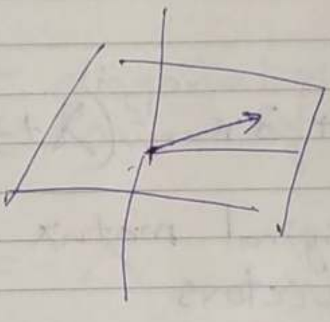
Linear Algebra (Gilbert Strang)
(Part 2)
Youtube

Lec-21

- Eigen-values & Eigen Vectors
- $\det[A - \lambda I] = 0$
- Trace $= \lambda_1 + \lambda_2 + \cdots + \lambda_n$

Ax parallel to x $\Rightarrow$ eigen-vectors

If A is singular, then $\lambda = 0$ is an eigen value



Any x in the plane:
$Px = x$ $\boxed{\lambda = 1}$

Any x $\perp^r$ to plane
$Px = 0 = 0x$ $\boxed{\lambda = 0}$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (permutation matrix)

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the eigen vector

$\lambda = 1$ $\qquad$ $\lambda = -1$

$\Downarrow$

$Ax = x$ $\qquad$ $Ax = -x$

fact: Sum of λ's = Sum of diagonals

How to find eigen vector for any A

$Ax = \lambda x$

$(A - \lambda I)x = 0$

x is non zero. Since non-zero x is in null space of "$A - \lambda I$", "$A - \lambda I$" must be singular

If $Ax = \lambda x$

then $(A + 3I)x = \lambda x + 3x = (\lambda + 3)x$

ie. adding I to original matrix does not change the eigen vectors

Rotation Matrix (Q)

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} \text{trace} &= 0 = \lambda_1 + \lambda_2 \\ \det(Q) &= 1 = \lambda_1 \lambda_2 \end{aligned}\right\} \Rightarrow \text{imaginary eigen value}$$

triangular Matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

eigen-values: 3, 3 (repeated)

for triangular matrix diagonal values are the eigen values.

$\lambda = 3$ & 3

find eig vec

$(A - \lambda I)x = 0$

$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}(x) = 0$

$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad X_2 =$ no sol^n that is independent to X_1

&

Lec-22

- diagonalizing a matrix $S^{-1}AS = \Lambda$
- Powers of A / eq^n $u_{k+1} = Au_k$

$S^{-1}AS = \Lambda$

$\downarrow$

Suppose, there are n ind. eigen vectors of A, put them in cols of S

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

$$= \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

$$= S\Lambda$$

$$AS = S\Lambda$$

$$S^{-1}AS = \Lambda$$ (provided S is invertible ie. there are n ind eigen vectors)

$$\Updownarrow$$

$$A = S\Lambda S^{-1}$$

A^2 can be easy computed from eigen-v

$\Rightarrow$ for A^2, eigen vector remains the same but, eigen values are squa

$$A^2 = (S\Lambda \underline{S^{-1})(S}\Lambda S^{-1})$$

$$= S\Lambda\Lambda S^{-1}$$

$$= S(\Lambda\Lambda)S^{-1}$$

$$\Rightarrow A^k = S\Lambda^k S^{-1}$$

Theorem

$A^k \to 0$ as $k \to \infty$

if all eigen values have to be less than 1, ie $|\lambda| < 1$

Solving difference eq^n.

$$u_k = Au_{k-1}$$

$$\Updownarrow$$

$$u_k = A^k u_0$$

note: only valid for 1-order difference eqⁿ

eigen vectors of A

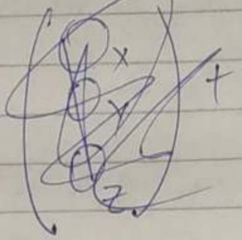
$$u_0 = C_1 x_1 + C_2 x_2 + \dots C_n x_n$$

$$A u_0 = C_1 A x_1 + C_2 A x_2 + \dots C_n A x_n$$

$$= C_1 \lambda_1 x_1 + C_2 \lambda_2 x_2 + \dots C_n \lambda_n x_n$$

$$u_k = A^k u_0 = C_1 \lambda_1^k x_1 + C_2 \lambda_2^k x_2 + \dots C_n \lambda^k x_n$$

trick to compute A^{100}



Solving Fibonnacci Series with this

$F(n) = F(n-1) + F(n-2)$

$F(n-1) = F(n-1)$

$$\begin{pmatrix} F(n) \\ F(n-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F(n-1) \\ F(n-2) \end{pmatrix}$$

$$u_n = A u_{n-1}$$

$$= A^{n-1} u_0$$ ⇒ solve with eigen property

Note: if A is singular one of the eigen value is '0'

lec-23

- Differential Eq^n $\frac{du}{dt} = Au$
- Exponential e^{At} of a matrix

Solve,

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$\Updownarrow$$

$$\begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \frac{d\bar{u}}{dt} = A\bar{u}$$

Eigen

$$\lambda_1 = -3 \qquad \lambda = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Sol^n

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$= C_1 e^{0t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

exponential of a matrix

$e^{At} =$ ~~$S\Lambda$~~ $S e^{\Lambda t} S^{-1}$

↳ Proved from next power series

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use of eigen

Power Series

$e^{At} = I$

$(I - At)^{-1}$

$=$

↳ $e^{At} = I$ ↓ SS

Lec-24

A

①
②
③

use of expns

Power Series — generalized to matrix

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots + (At)^n + \cdots$$

$$\to e^{At} = \underset{\substack{\downarrow \\ SS^{-1}}}{I} + S\Lambda S^{-1}t + S\Lambda^2 S^{-1}\frac{t^2}{2!} + \cdots$$

$$= S\left[I + \Lambda t + \Lambda^2 \frac{t}{2!} + \cdots \right] S^{-1}$$

$$= S e^{\Lambda t} S^{-1}$$

Lec-24

- Markov Matrices
- Steady State
- Fourier series & Projections.

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

① All entries are +ve
② Col-wise sum is 1

③ a) $\lambda = 1$ is always an eigen-value of markov matrix
b) All other eigen-values $|\lambda| < 1$

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \dots + c_n\lambda_n$

Steady state, $k \to \infty$

1 eigen value is ①

$\therefore u_{\text{Steady State}} = c_1 (1) \cdot x_1$

Why $\lambda = 1$ is a eigen value of markov.

$$A = \begin{bmatrix} 0.1 & +0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}$$

$\Rightarrow$

sum of 2 code of col = 3rd col

ie. sum of cols is 0

$\Downarrow$

Singular

$\Downarrow$

$\lambda = 1$

Can be easyly proved for a general matrix

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relation between eigen values of A & A^T

$\Downarrow$

they are the same

Application of Markov : Population movement modelling

Projection with orthonormal basis

$q_1, q_2 \cdots q_n$

any $\bar{V} = x_1 q_1 + x_2 q_2 + \cdots x_n q_n$

$\hookrightarrow$ expanding the vector in terms of basis

way 1 to find coit.

$$q_1^T \bar{V} = x_1 q_1^T q_1 + x_2 q_1^T q_2 + \cdots x_n q_1^T q_n$$

$$q_1^T V = x_1 + 0 \cdots + 0$$

$\therefore x_1$

way 2 - to find coif

$$\bar{V} = x_1 q_1 + x_2 q_2 \cdots x_n q_n$$

$$= \begin{pmatrix} q_1 & q_2 & q_2 & \cdots & q_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\bar{V} = Q \bar{X}$$

$$\therefore \bar{X} = Q^{-1} \bar{V} = \underbrace{Q^T V}_{\text{orthonormal}}$$

Fourier Series

$$f(X) = a_0 + a_1 \cos X + b_1 \sin X + a_2 \cos X + b_2 \sin X + \cdots \infty$$

vectors

$$V^T W = v_1 w_1 + v_2 w_2 + \cdots v_n w_n$$

f^n

$$f^T g = \int f(x) g(x) \, dx$$

Lec-24b (review)

- Orthogonality (Q) $\Rightarrow Q^TQ = I$
  Projections, least Square
  Gram-Schmidt

- det A
  Properties
  big formula n! term (+, -)
  co-factor formula (A^{-1})

- Eigen values & Eigen Vectors
  $Ax = \lambda x$
  $\det(A - \lambda I) = 0$
  diagonalize $S^{-1}AS = \Lambda$
  Powers of A

note:

1) A is invertivle ⟺ none of λs are 0
2) product of λs is the det (A)
3) $\det A^{-1} = \left(\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_2}\right) \cdots \left(\frac{1}{\lambda_n}\right)$
4) $(A + I)$ will have eigen values as $(\lambda_1 + 1), (\lambda_1 + \lambda_2)$, + ...
5) trace(A) = sum of all eigen values

Lec 25

- Symmetric matrix (Eigen val / vec)
- +ve definite matrices

$A = A^T$

↓

eigen values are real ✓

eigen vectors are orthogonal

Usual $A = S \Lambda S^{-1}$

Symmetric $A = Q \Lambda Q^{-1} = Q \Lambda Q^{T}$

Spectral theorem

Why real eigen values for a real Symmetric matrix ??

$Ax = \lambda x$ — ① $\longrightarrow \boxed{\bar{x}^T A x = \lambda \bar{x}^T x}$ — Ⓐ

take conjugate

$A^* x^* = \lambda^* x^*$ — ⓐ

$\therefore A x^* = \lambda^* x^*$ $[\because A \text{ is real}] \longrightarrow$

transpose

$(x^*)^T A^T = \lambda^*(x^*)^T$ — ② $\Rightarrow (x^*)^T A = \lambda^*(x^*)^T$

[multiplying x on both sides]

$\Rightarrow \boxed{(x^*)^T A x = \bar{x}^T \bar{\lambda} x}$ — Ⓑ

mul $(x^*)^T$ on both sides of ⓐ

$\underline{(x^*)^T A x^*} = \lambda^* (x^*)^T x^*$

from ②

$\lambda (x^*)^T x^* = \lambda^* (x^*)^T (x^*)$

if $(x^*)^T x$ is not zero.

$\Rightarrow \lambda = \lambda^* \Rightarrow \lambda$ is real.

Compare A, B

$(\bar{x}^T x)\lambda = \bar{\lambda}(\bar{x}^T x)$

$\therefore \lambda$ is real.

#

every Symmetric is a combination ⊥ projection matrices

$$A = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 q_1^T \\ \lambda_2 q_2^T \\ \vdots \\ \lambda_n q_n^T \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots \lambda_n q_n q_n^T$$

$=$

for symmetric matrices ($A=A^T$)

① Signs of pivots are same as signs of eigen values.

② # of +ve pivots = # of +ve eigen values.
(use full to compute eigen values.

③ for (A) symmetric matrix
product of pivots = product of eigen values
(because they both eq. the determinant)

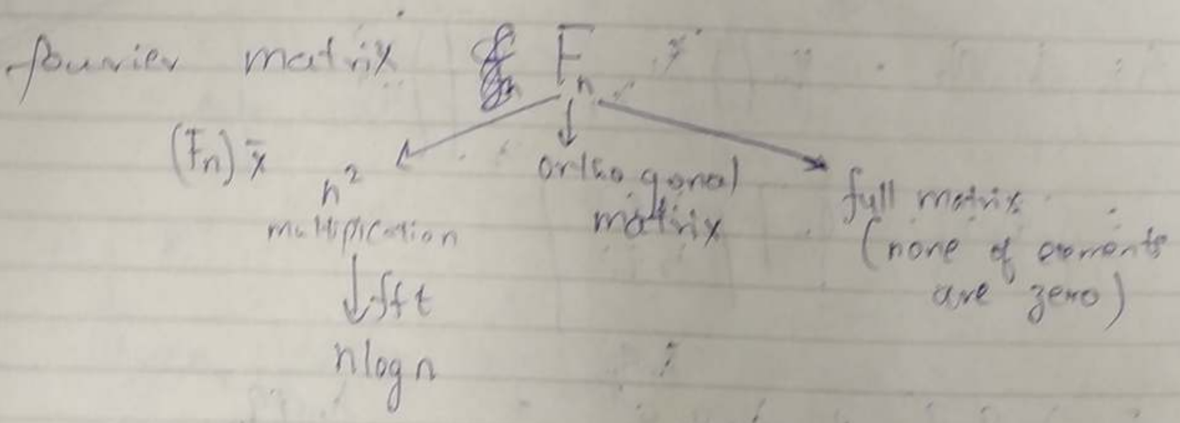
Positive definite symmetric Matrix

② ① all eigen values are +ve (the all pivots are

also
all sub-dets | of +ve definite is +ve

Lec-26

- complex vectors & matrices
- inner product
- discrete fourier transform
- FAst transform (FFT)



I) Changing def^n of inner product (for complex vec)

$$\underset{\substack{\text{has} \\ \text{complex} \\ \text{numbers}}}{Z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} \qquad Z \in \mathbb{C}^n \ (\text{not } \mathbb{R}^n$$

$$\Rightarrow \|Z\|^2 \triangleq \underset{\text{conj}(Z)}{\bar{Z}^T} Z \rightarrow |z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots + |z_n|^2$$

Z^H : Hermitian $\Leftrightarrow$ conj & transpose a matrix

Summary:

inner product of real vector: $y^T y$

of complex vec: $Z^H Z := \bar{Z}^T Z$

II) Changing def^n of Symmetric (for complex mat)

Symmetric real matrix : $A^T = A$

Symmetric for complex : $\bar{A}^T = A$

$$\Updownarrow$$

$$A^H = A$$

$$\text{eg: } \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} \leftarrow \text{hermitian matrices}$$

II) Orthogonality (fr) (unitary matrix ie ⊥r unit in $\mathbb{C}^n$) (for complex mat)

real

$q_1, q_2 \cdots q_n$

$$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$Q^T Q = I$$

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

complex

$q_1, q_2 \cdots q_n$

$$\bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\bar{Q}^T Q = I$$

$$\Updownarrow$$

$$Q^H Q = I$$

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & & W^{N-1} \\ \vdots & W^2 & W^4 & & W^{2(n-1)} \\ & \vdots & \vdots & & \vdots \\ 1 & W^{n-1} & W^{2(n-1)} & & W^{(n-1)^2} \end{bmatrix}$$

$(F_n)_{ij}$ $\quad i, j = 0, \cdots n-1$

$(F_n)_{ij} = (\omega)^{ij}$

where $\omega = e^{i 2\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

eg: $n = 4$ $\quad \omega = e^{2\pi i/4} = i$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$F_4^H F_4 = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 \\ 0 & 0 & d \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & d \end{pmatrix}$$

$$\tilde{F}_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$\nearrow$ length of vector

fast f^T

Connection between F_n & $F_{n/2}$

ie F_4 & F_2 ; F_{64} & F_{32}

~~$\begin{bmatrix} F_{64} \end{bmatrix} = 0 \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}$~~

64x64 mat

$$F_{64} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} & 1 & & \\ & & 1 & \cdots 1 \\ 1 & 1 & \ddots & \\ 1 & & & 1 \end{bmatrix}$$

odds & evens mat permutation

Permutation

diagonal matrix

$$\begin{bmatrix} 1 & & & & 0 \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \omega^3 & \\ 0 & & & & \ddots \; \omega^n \end{bmatrix}$$

$$\begin{bmatrix} \text{even numbered} \\ \text{Components} \\ \text{odd numbered} \\ \text{Components} \end{bmatrix}$$

$$W_n = e^{i\frac{2\pi}{n}} \quad ; \quad (W_n)^2 = \left(e^{i\frac{2\pi}{n}}\right)^2 = e^{i\frac{2\pi}{n/2}} = W_{n/2}$$

Lec-27

- +ve definite matrix
- Te sts for minimum $(x^T A x > 0)$
- Ellipsoid in $\mathbb{R}^n$

+ve definite

Matrix is symmetric (implicit assumption)

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

① $\lambda_1 > 0$; $\lambda_2 > 0$ (eigen val test)

② $a > 0$ $\quad ac - b^2 > 0$ (determinants)

③ $a > 0$ $\quad \frac{ac - b^2}{a} > 0$ (pivots)

④ * $x^T A x > 0$

+ve-Semi-definite

$$\lambda_1, \lambda_2 \geq 0$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & \boxed{18} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix}$$

$$2x_1^2 + 12x_1x_2 + 18x_2^2$$

$$2(x_1 + 3x_2)^2$$

Graphs of $f(x,y) = \bar{X}^T A \overset{\begin{bmatrix} x \\ y \end{bmatrix}}{\bar{X}} = ax^2 + 2bxy + cy^2$

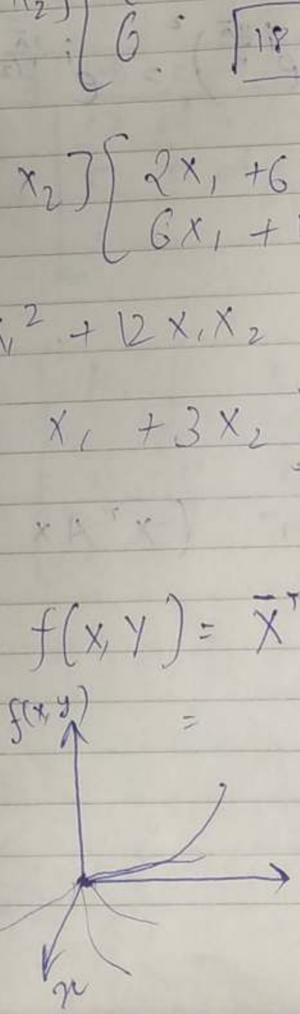
=

$2x^2 + 12xy + 7y^2$

↑

non- +ve definite

(has saddle point)



$$[x, y]\begin{bmatrix} 2 & 6 \\ 6 & \boxed{20} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Downarrow$$

$$2x^2 + 12xy + 20y^2$$

$$\Downarrow$$

+ve every where ~~exc~~ except at minimum pt.

Calculus:

Min ~ $\frac{d^2u}{\partial x^2} > 0$, ~~AND~~, deriv = 0

MIN ~ Matrix of 2nd deriv. is +ve def.

$f(x_1, x_2 \cdots x_n)$

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matrix of 2nd derivatives. $\begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{bmatrix}$

eg: 3×3

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is it +ve def ? | whats $z = x^T Ax$??

⇒ yes (all pivots are +ve)

1×1 ↑ ; 2×2 ↑ ; 3×3 ↑

det : 2 ; 3 ; 4

Pivots : 2 ; $\frac{3}{2}$; $\frac{4}{3}$

(Product of pivots = det of matrix)

eigenvalues : $2-\sqrt{2}$, 2, $2+\sqrt{2}$

(Gilbert guessed it)

Lec-28

- A^TA is +ve definite
- Similar matrices A, B / Jordan form

  $B = M^{-1}AM$
- Jordan form / Jordan block

* if a matrix is +ve definite then A^{-1} is also +ve definite
($\because$ eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots \frac{1}{\lambda_n}$)

* if A & B are +ve definite;
what about A + B ?
$\Rightarrow$ yes

proof: $x^T(A+B)x$

$= x^TAx + x^TBx$

$\Rightarrow$ +ve definite

* A is rectangular $m \times n$

$A^T A \Rightarrow$ square, Symmetric, +ve definite

Proof:

$$x^T (A^T A) x$$
$$= (x^T A^T)(Ax)$$
$$= (Ax)^T Ax = (\text{length})^2 \geq 0$$

$\Rightarrow$ +ve definite.
(+ve semi definite)

Condition for $Ax \neq 0$ (ie no null space)
$\longrightarrow$ $A^T A$ has independent cols
ie rank is n

$\underset{m \times n}{A}$ & $\underset{m \times n}{B}$ are similar

means for some matrix M,

$$B = M^{-1} A M$$

$$Ax = \lambda x$$
$$\Rightarrow (\underbrace{M^{-1} A M}_{B}) M^{-1} x = \lambda M^{-1} x$$
$$B\, M^{-1} x = \lambda M^{-1} x$$

$\therefore$ λ is an e.v of B

example: Here A is similar to Λ and M is e.v Matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \underset{\substack{\uparrow \\ \text{arbitary} \\ M}}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \underset{\substack{\uparrow \\ B}}{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}$$

Note: 2 similar matrices have Same eigen values.

A & B are Similar

A: no repeated eigen → $J = \Lambda$; repeated eigen (Jordan)

If $\lambda_1 = \lambda_2$, then it's not possible to diagonalize

Bad possibility: 2 eigen values are the Same

$\lambda_1 = \lambda_2 = 4$

one family : $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow$ 1 member only

2nd family : $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \Rightarrow$ larger family

Jordan form

eg: $L' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\lambda = 0, 0, 0, 0$

will have 2 eig-vectors

$\dim(N(A)) = 4 - \underset{\uparrow \text{ \# of ind cols.}}{2} = 2$

eg 2: $M' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\lambda = 0, 0, 0, 0$

$\dim(N(A)) = 2$

But: L', M' are not similar.

Jordan block

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ 0 & & \lambda_i & 1 \\ & & & \ddots \end{bmatrix}$$

2nd diag has "1"

1st diag has repeated eigen vals.

has 1 eigen-vector

- L' has a block of 3x3, 1x1
- M' has 2 2x2 block

Jordan's theorem

every sq matrix A is similar to a Jordan matrix J.

$$J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_d} \end{bmatrix}$$

block = # of eigen vectors.

note:

Jordan form is a generalization of eigen-vec-vals.

in matlab

[V, J] = jordan(A)

If A has no repeated value J is same as Λ, ~~V is same as eigen vectors~~.

related to: Schur decomposition

(see wiki)

Lec-29

- Singular Value Decomposition (SVD)

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

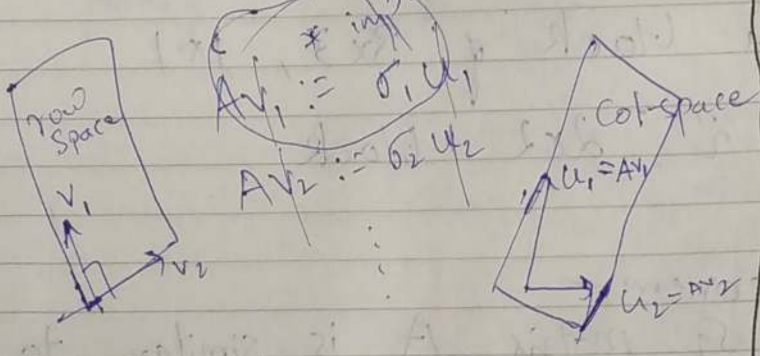
Any Matrix ↗ A; ↑ Σ diagonal / U V are orthogonal

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_r u_r \end{bmatrix}$$

* imp?

$$Av_1 = \sigma_1 u_1$$

$$Av_2 = \sigma_2 u_2$$



A typical vector v_1 in row space is taken over u_1 in Col space.

v_2 orthogonal to v_1 in Row space is taken over orthogonal basis in Col space [

σ_1, σ_2 will be multiple of unit vector $(u_1, u_2 \ldots)$ [u_1, v_1 are unit vector]

$$\underset{m\times n}{A}\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix}\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \; \sigma_r \end{bmatrix}$$

basis vector of row space

basis vector for col space

$$AV = U\Sigma$$

eg $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$; look for

↑ any general mat

v_1, v_2 orthonormal in row space $\mathbb{R}^2$

u_1, u_2 orthonormal in col space $\mathbb{R}^2$

$\sigma_1 > 0$; $\sigma_2 > 0$

null Spaces are no problem here

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r & 0 & 0 \end{bmatrix}\begin{bmatrix} \sigma_1 & \cdots \end{bmatrix}$$

imp note: eigen vectors of Symmetric matrices are orthogonal!

Any matrix

$$AV = U\Sigma \Rightarrow A = U\Sigma V^{-1} = U\Sigma V^T$$

$\left[\because V \text{ is orthonormal mat} \therefore V^{-1} = V^T\right]$

$$A^TA = (V\Sigma^T U^T)(U\Sigma V^T)$$

$$= V\Sigma^T\Sigma V^T$$

$$= V\begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix}V^T$$

.... can be computed with eigen-decomp. of A^TA

$$AA = U\begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix}U$$

..... can be computed with eigen-decomp of AA^T

eg ① $A^TA = \begin{bmatrix}4 & -3\\ 4 & 3\end{bmatrix}\begin{bmatrix}4 & 4\\ -3 & 3\end{bmatrix} = \begin{bmatrix}25 & 7\\ 7 & 25\end{bmatrix}$

$\therefore$ e.v$_1$ = $\begin{bmatrix}1\\1\end{bmatrix}$ $\therefore$ Normalize e.v = $\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \Rightarrow A\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = 32\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$\begin{bmatrix}1\\-1\end{bmatrix}$ $\therefore$ " " = $\begin{bmatrix}\frac{1}{\sqrt{2}}\\-1/\sqrt{2}\end{bmatrix} \Rightarrow A\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} = 18\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$

U

$$A = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}\sqrt{32} & 0\\ 0 & \sqrt{18}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

conclusion

$$A = \underset{\substack{\uparrow\\ \text{basis of}\\ \text{col-space}}}{U}\ \Sigma\ \underset{\substack{\nwarrow\\ \text{basis of}\\ \text{row-space}}}{V^T}$$

for rank r

$v_1, \dots v_r$ orthonormal basis of col space

$u_1, \dots u_r$ orthonormal basis of row space

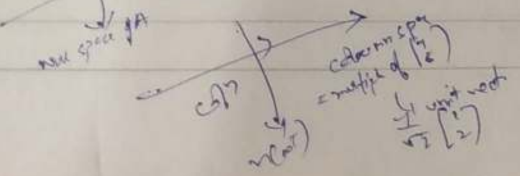
$v_{r+1}, \dots v_n$ orthonormal basis of null space

$u_{r+1}, \dots u_m$ orthonormal basis of $n(A^T)$

Eg (2): $A = \begin{bmatrix}4 & 3\\ 8 & 6\end{bmatrix}$ $\therefore$ Row space is a vector passing (4, 3)

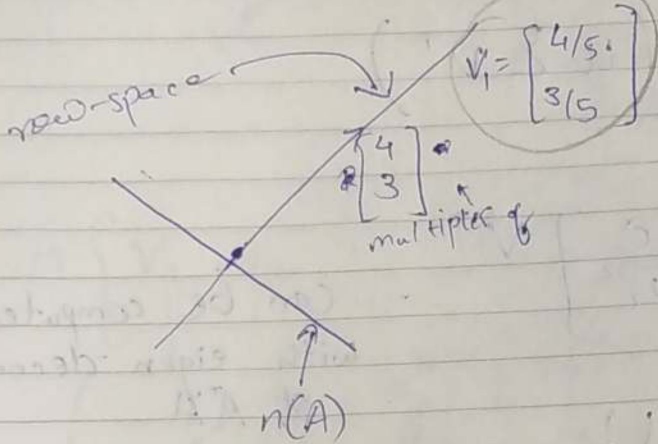
$$\begin{bmatrix}4 & 3\\ 8 & 6\end{bmatrix} = \underset{U}{\begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5}\\ 2/\sqrt{5} & -1/\sqrt{5}\end{bmatrix}}\underset{\Sigma}{\begin{bmatrix}\sqrt{125} & 0\\ 0 & 0\end{bmatrix}}\underset{V^T}{\begin{bmatrix}.8 & .6\\ .6 & -.8\end{bmatrix}}$$

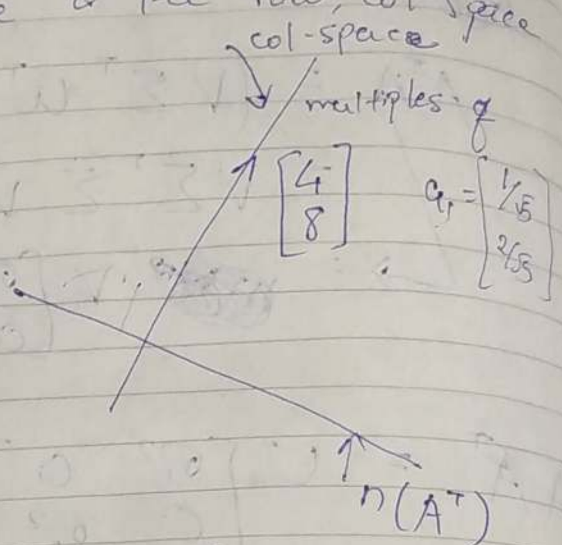
(125) ca from A^TA



eg: $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

note: has a null space & 1-d row, col-space





$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & \bigcirc \\ 2/\sqrt{5} & \end{bmatrix} \begin{bmatrix} \overline{\sqrt{\text{eigen val of } A^TA}} \\ \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 \\ \bigcirc & \end{bmatrix}$$

n(A) — rank-1 matrix — $n(A^T)$

Lec-30

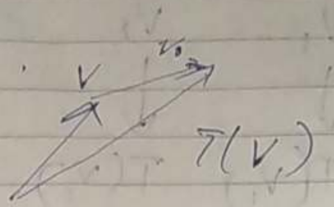
- Linear transformation T

a. $T(u+v) = T(u) + T(v)$

b. $T(cu) = c\,T(u)$

eg:- Shift by v_0

⇒ not a linear transform



note $T(0) = 0$ for a linear transform

eg:

$T(v) = \|v\|$ $\quad T: R^3 \rightarrow R^1$

⇒ not linear

$T(-2v) \neq -2T(v)$ but $= 2T(v)$

thus, not linear

eg3:

Imp. matrix A

$T(v) = Av$

→ linear

$T(v+w) = A(v+w) = Av + Aw = T(v) + T(w)$

$T(cv) = A(cv) = cAv = cT(v)$

* All linear transformation can be represented as matrix multplication

$$\begin{tikzcd} V_1 \arrow[d] & V_2 \arrow[d] \\ T(V_1) & T(V_2) \end{tikzcd}$$

having known the output for v_1, v_2 we can tell the transformation outcome of all linear combinations of v_1 & v_2.

if we know the transform output of every basis $v_1, v_2, \dots, v_n$, we known the outcome to any $v \in \text{span}(\underset{n \text{ basis of space}}{v_1, v_2, \dots, v_n})$.

rule to find $\overset{\text{transformation}}{A}$. ([illegible]) ; Given basis

1^{st} col of A ~~[illegible]~~ write linear transform to $T(V_1)$

eg:- linear transformation that takes derivative

input : $c_1 + c_2 x + c_3 x^2$ basis $1, x, x^2$

Output: (deri) $c_2 + 2c_3 x$ basis $1, x$

think abt output of basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\0\end{pmatrix} ; \begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix} ; \begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\2\end{pmatrix}$$

$$\underset{\text{output}}{\begin{pmatrix}c_2\\2c_3\end{pmatrix}_{2\times1}} = \underset{\text{derivative}}{\begin{pmatrix}0 & 1 & 0\\0 & 0 & 2\end{pmatrix}_{2\times3}} \underset{\text{input}}{\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}_{3\times1}}$$

Lec-31

- Change of basis
- Compression of Image
- Transformation ⟷ Matrix

Standard basis -

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\\vdots\\0\end{bmatrix} \cdots\cdots \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

Better basis -

$$\begin{bmatrix}1\\1\\1\\1\\\vdots\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix} \qquad \begin{bmatrix}1\\-1\\1\\-1\\\vdots\\1\end{bmatrix}$$

$\uparrow$ half ones, half -1

Fourier basis - (8x8)

$$\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}, \begin{bmatrix}1\\w\\w^2\\\vdots\\w^{n-1}\end{bmatrix}, \begin{bmatrix}1\\w^2\\w^4\\\vdots\end{bmatrix}, \cdots\cdots$$

image compression example.

- Change of basis

Orthogonal matrices has $|\lambda_i| = 1$

Change of basis

$$\underset{\substack{\uparrow \\ \text{in std} \\ \text{basis}}}{\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_n \end{bmatrix}} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdots \overset{\text{wavelet basis}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} c_8$$

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} | & | & | & | & | & | & | & | \end{bmatrix} \underset{\substack{\uparrow \\ \text{coif in wavelet} \\ \text{basis}}}{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}}$$

$$\bar{P} = W\bar{C}$$

$$\therefore \bar{C} = W^{-1}P$$

good basis will have fast inverse

note W is orthonormal

$$\therefore W^{-1} = W^T$$

note: If we change the basis, every transformation matrix changes

$$B = W^{-1}\underset{\substack{\uparrow \\ \text{org transform}}}{A}W \leftarrow \text{basis matrix}$$

A & B are similar.

note: Symmetric, Anti-Symmetric, orthogonal Mat have orthogonal eigen-vectors. In particular when $A^TA = AA^T$ then eigen vectors are orthogonal.

I know a transform completely, if I know what it does to the basis.

$T(v_1), T(v_2) \dots, T(v_n)$

$\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}$

Quiz Review - 3

- λ & x (eigen)
- $du/dt = Au$, e^{At}
- $A = A^T$ (Symmetric are +ve definite)
  (real eigen value)
  (orthogonal eigen vectors) $Q\Lambda Q^T$
- Similar Matrix
- SVD

note: for diagonizable, we need enough eigen-vectors. dont care about eigen-vals

eg:-

$\lambda_1 = 0 \quad \lambda_2 = c \quad \lambda_3 = 2$

$\downarrow \quad \downarrow \quad \downarrow$

$\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}1\\-1\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\-2\end{bmatrix}$

Q) Is it diagonalization?

yes, for all c.

Q) For what values of C it is Symmetric for real C, note: if eigen-~~vectors~~ vals of A are λ_i then eigenvals of $A/2$ will be

∴ 0

Q) +ve definite ?

~~for real +ve c~~

∴ never cause another $\lambda = 0$ ∴

but can be +ve semi-definite for real

+ve Q. C

Q) Can it be markov matrix ?

no. all eigen vales $|\lambda| < 1$

Q) Could $A/2$ be a projection matrix

for a matrix to be projection

matrix $P^2 = P \Rightarrow \lambda$ is either

0s or 1s

note: Orthogonal Matrix has $|\lambda_i| = 1$

Proof:

$$Qx = \lambda x \quad \& \quad \|Qx\| = \|\lambda x\|$$

$$\Updownarrow$$

$$\|Q\| \, \|x\| = \|\lambda\| \, \|x\|$$

$$\downarrow$$

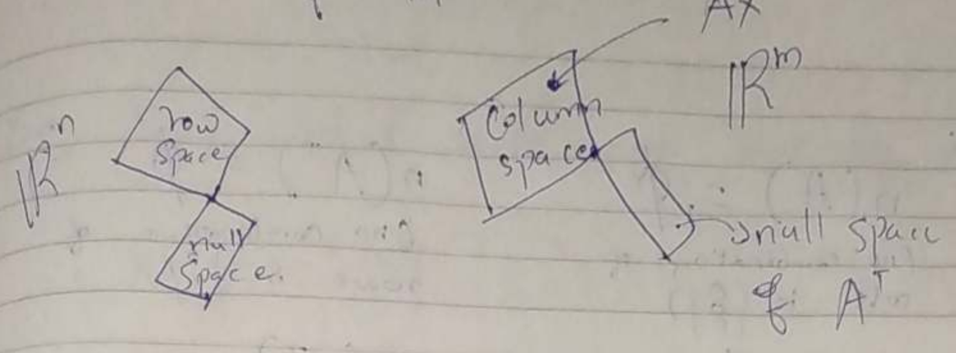
I (property of orthogonal mat.

$$\therefore |\lambda| = 1$$

Lec-33

- left, right Inverses
- Pseudo-inverses

of $A_{m\times n}\, x$



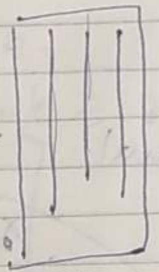
① $AA^{-1} = I = A^{-1}A$ (no-matter which side we write (A^{-1}) we get I).

$r=m=n$
for full rank

\# (cols are ind.)
full col rank (left inverse)

$r = n < m$ (# of cols)

\# 0 or 1 soln to $Ax=b$



\# What about $A^T A$
→ rank of A^TA will be full
hence A^TA is invertible.

Note: $\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}}\, A_{m\times n} = I_{n\times n}$

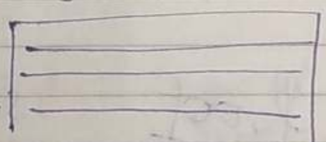
3x3
(full rank
∴ inv exist)

\# $\underset{n\times m}{A^{-1}_{left}}\ \underset{m\times n}{A} = I_{n\times n}$

\# $\cancel{\underset{m\times n}{A}\ \underset{n\times m}{A^{-1}_{left}} = I_{m\times m}}$

---

full row rank (right inverse)
$r = m < n$, $n(A^T) = \{0\}$, ∞ soln to $Ax=b$



What about AA^T

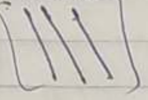
$A\left(A^T(AA^T)^{-1}\right) = I$

$A\ \ A^{-1}_{right} = I$

* rectangular can not have both inverses.

full rank Case - I : $n(A) = \phi$ (no combinations of cols is $\begin{bmatrix}0\\0\end{bmatrix}$) $n(A^T) = \phi$ (no combinations of rows is $\begin{bmatrix}0\\0\\0\end{bmatrix}$)

Case II



$n(A) = \phi$ $n(A^T) = \underline{\quad}$

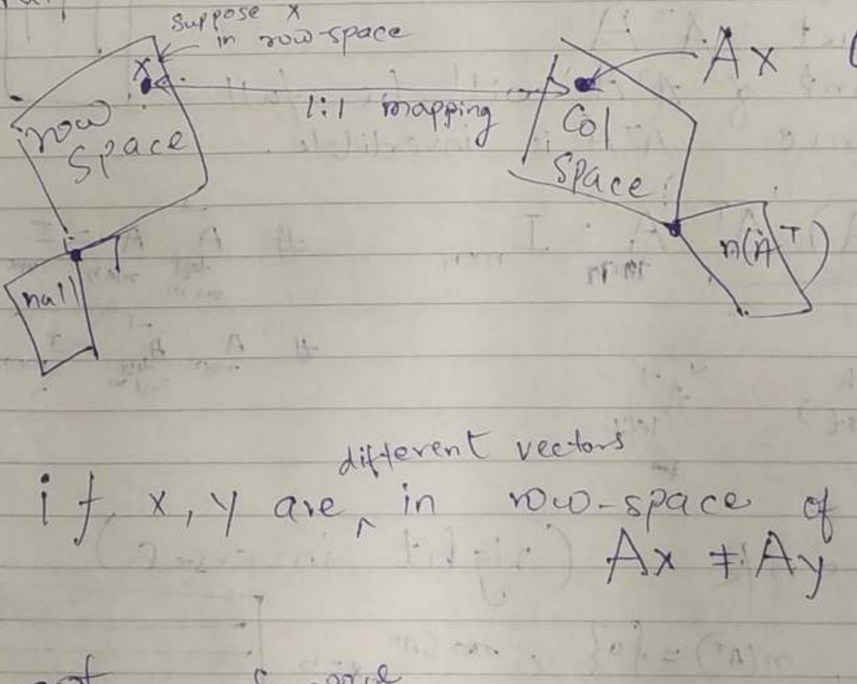
Case III :

$n(A) = \underline{\quad}$ $n(A^T) = \phi$

Case IV

$n(A) = \underline{\quad}$ $n(A^T) = \underline{\quad}$

General $r < m < n$



Ax will be in colspace

* if x, y are different vectors in row-space of A then prove $Ax \neq Ay$

Proof suppose

$$Ax = Ay$$

$$\therefore A(x-y) = 0$$

this mean $(x-y)$ is in null-space of A but by given assumption x, y are in rowspace

So, x - y must also be in row-space

$$\therefore x - y = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Find the pseudo-inverse A^+.

1) start from SVD

$$A = U \Sigma V^T$$

$$\Sigma = \begin{bmatrix} \sigma_1 & \ddots & 0 & \\ & 0 & \sigma_r & 0 \end{bmatrix} \; m$$

n

rank $\equiv r$

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & \cdot & 0 & \\ 0 & & 1/\sigma_r & 0 \end{bmatrix}_{m \times n}$$

$$\underset{m\times m}{\Sigma^+ \Sigma} = \begin{bmatrix} 1 & & 0 & \\ 0 & 1 & \cdot & \\ & & 1 & 0 \cdots 0 \end{bmatrix}$$

$$\Sigma \Sigma^+ = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & & 0 \end{bmatrix}_{n \times n}$$

(r - Ones)

$$A = U \Sigma V^T$$

$$A^+ = V \Sigma^+ U^T$$ (Since, U, V are orthogonal matrix)

Lec-34 Course Review

(Best way is examples)

Q) $\overset{3\times n}{A}x = \begin{bmatrix}1\\0\\0\end{bmatrix}$ has no sol^n ——— ①

$Ax = \begin{bmatrix}0\\1\\0\end{bmatrix}$ has unique sol^n ——— ②

find # rows, cols, rank of A; give an example

Ans)
a) # rows = 3

note: if there are no-sol^n cases implies r is below m (# of rows)

eg: $2x + 3y + 4z = 5$
$4x + 6y + 8z = 11$ ⇒ no sol^n
$x + y + z = 1$

$$\begin{pmatrix}2 & 3 & 4\\4 & 6 & 8\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\11\\1\end{pmatrix}$$

rows are → not ind, hence no sol^n

⇒ rank will be 2, or 1.

b) has unique sol^n ⇒ there is no null-space

$m = 3 > n = r$

∴ eg-1 $A = \begin{bmatrix}0\\1\\0\end{bmatrix}$

eg-2 $\begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}$